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THE PLANE SOLUTION FOR BENDING OF JOINED DISSIMILAR ELASTIC SEMI-INFINITE STRIPS

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Abstract—The problem is reduced to a system of two singular integral equations for determining the interface slope and shear stress. The dominant part of the system is analyzed to determine the order of the stress singularity and its dependence on the elastic constants. After removing logarithmic singularities from the right hand sides we solve these equations numerically for several chosen composites and the interface slope and traction are exhibited graphically. The solution should be relevant in studying adhesive joints by means of a bending test.

INTRODUCTION

In [1] the solution was given for joined semi-infinite elastic strips in tension. In [2] the end problem for a semi-infinite strip was solved for point-wise traction, displacement or mixed end conditions. This solution was applied there to several specific problems including extension, bending, and flexure for fixed end; compression against a rigid circular cylinder (smooth and rough); and the pointwise normal traction problems of extension and bending with cosine and sine distributions, respectively. The singular integral equation obtained in [2] for the bending problem considered there had the added complication of a logarithmically singular function in its right hand side. This difficulty was not resolved in [2] and no numerical results were given there for the bending case.

Here the antisymmetric solution given in [2] is used in the manner of [1] to solve the bending problem for the bimaterial strip. The problem is reduced to a system of two singular integral equations. Then the dependence of the order of the stress singularity, at the point of intersection of the material interface and lateral boundary, on the material parameters is extracted from the integral equations. The difficulty of logarithmic singularities in the right hand sides of the equations occurs here also, but it is successfully dealt with by use of a special technique. Finally, the integral equations are solved numerically and the bond tractions and displacements are computed and graphically illustrated.

FORMULATION OF THE PROBLEM

Consider the two isotropic homogeneous elastic semi-infinite strips of width $2h$ characterized by the elastic constants μ'', ν'' and μ', ν' (Fig. 1a). Let the strips be joined at $x_2 = 0$, free of traction on $|x_1| = h$, and loaded far from the interface by a linear normal stress in the x_2-direction which produces the resultant couple $-M$ on any plane of fixed x_2. Accordingly, we wish to find the two-dimensional stress and displacement fields $\hat{S}'' = \{\hat{\tau}'', \hat{\mathbf{u}}''\}$, $\hat{S}' = \{\hat{\tau}', \hat{\mathbf{u}}'\}$, that satisfy the appropriate elasticity equations for $x_2 > 0$, $x_2 < 0$, respectively, and meet the boundary conditions

$$\begin{aligned}
&\hat{\tau}''_{12} \to 0, \quad \hat{\tau}''_{22} \to -(3M/2h^3)x_1; \quad |x_1| < h, \quad x_2 \to \infty,\\
&\hat{\tau}''_{11} = \hat{\tau}''_{12} = 0; \quad |x_1| = h, \quad 0 < x_2 < \infty,\\
&\hat{\tau}'_{11} = \hat{\tau}'_{12} = 0; \quad |x_1| = h, \quad -\infty < x_2 < 0,\\
&\hat{\tau}'_{12} \to 0, \quad \hat{\tau}'_{22} \to -(3M/2h^3)x_1; \quad |x_1| < h, \quad x_2 \to -\infty,
\end{aligned} \tag{1}$$

as well as the interface continuity conditions

$$\hat{u}''_1 = \hat{u}'_1, \quad \hat{u}''_2 = \hat{u}'_2, \quad \hat{\tau}''_{12} = \hat{\tau}'_{12}, \quad \hat{\tau}''_{22} = \hat{\tau}'_{22}; \quad |x_1| < h, \quad x_2 = 0. \tag{2}$$

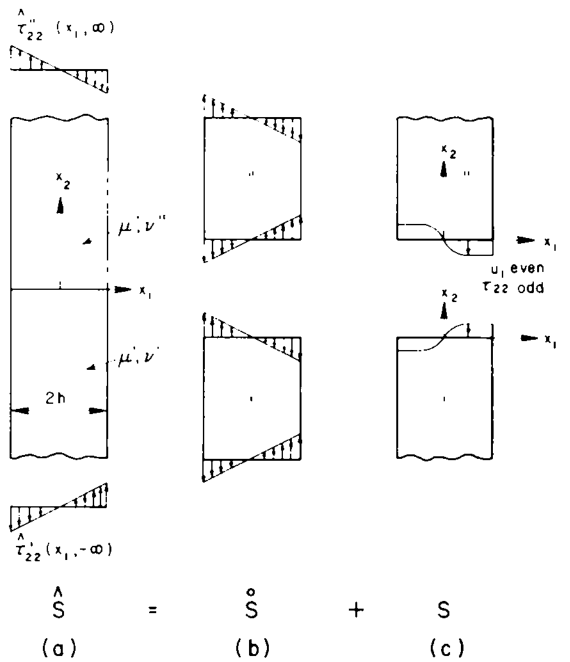

Fig. 1. Superposition of simple bending and residual solution.

AUXILIARY SEMI-INFINITE STRIP SOLUTIONS

In order to obtain the solution $\hat{S}''$, $\hat{S}'$ satisfying (1), (2), the solution depicted in Figs. 1b, c will be superposed. The solutions $\mathring{S}''$, $\mathring{S}'$ in Fig. 1b are the simple bending solutions given by†

$$\mathring{S}: \mathring{u}_1 = (3M/8h^3\mu)[\nu(x_1^2 - h^2) + (1-\nu)x_2^2]$$

$$\mathring{u}_2 = -(3M/4h^3\mu)(1-\nu)x_1x_2, \quad \mathring{\tau}_{11} = \mathring{\tau}_{12} = 0, \quad \mathring{\tau}_{22} = -(3M/2h^3)x_1, \tag{3}$$

with (′) or (″) attached to μ, ν, $\mathring{u}_\alpha$, $\mathring{\tau}_{\alpha\beta}$ for $\mathring{S}''$ or $\mathring{S}'$. The semi-infinite strip solutions depicted in Fig. 1c can be obtained from the antisymmetric solution in [2]. For S' this solution appears as

$$\sqrt{\left(\frac{\pi}{2}\right)}\, u'_1(x_1, x_2) = -\int_0^\infty \{[f'(\eta) - (1-2\nu')g'(\eta)]\cosh(\eta x_1) + \eta x_1 g'(\eta)\sinh(\eta x_1)\}\eta^{-1}\sin(\eta x_2)\,\mathrm{d}\eta$$

$$+\int_0^\infty \{-sx_2\phi'(s) + [-2+2\nu' - sx_2]s\omega'(s)\}s^{-1}\exp(sx_2)\cos(sx_1)\,\mathrm{d}s,$$

$$\sqrt{\left(\frac{\pi}{2}\right)}\, u'_2(x_1, x_2) = -\int_0^\infty \{[f'(\eta) + 2(1-\nu')g'(\eta)]\sinh(\eta x_1)$$

$$+ \eta x_1 g'(\eta)\cosh(\eta x_1)\}\eta^{-1}\cos(\eta x_2)\,\mathrm{d}\eta$$

$$+\int_0^\infty \{[3-4\nu' - sx_2]\phi'(s) + [1-2\nu' - sx_2]s\omega'(s)\}s^{-1}\exp(sx_2)\sin(sx_1)\,\mathrm{d}s,$$

$$\frac{1}{2\mu'}\sqrt{\left(\frac{\pi}{2}\right)}\,\tau'_{11} = -\int_0^\infty [f'(\eta)\sinh(\eta x_1) + \eta x_1 g'(\eta)\cosh(\eta x_1)]\sin(\eta x_2)\,\mathrm{d}\eta$$

$$+\int_0^\infty \{[2\nu' + sx_2]\phi'(s) + (2 + sx_2]s\omega'(s)\}\exp(sx_2)\sin(sx_1)\,\mathrm{d}s, \tag{4}$$

$$\frac{1}{2\mu'}\sqrt{\left(\frac{\pi}{2}\right)}\,\tau'_{22} = \int_0^\infty \{[f'(\eta) + 2g'(\eta)]\sinh(\eta x_1) + \eta x_1 g'(\eta)\cosh(\eta x_1)\}\sin(\eta x_2)\,\mathrm{d}\eta$$

$$+\int_0^\infty \{[2-2\nu' - sx_2]\phi'(s) - (sx_2)s\omega'(s)\}\exp(sx_2)\sin(sx_1)\,\mathrm{d}s,$$

†The solution will be derived here for the plane strain determination of Poisson's ratios. The generalized plane stress interpretation follows from the appropriate change in Poisson's ratios.

$$\frac{1}{2\mu'}\sqrt{\left(\frac{\pi}{2}\right)}\,\tau'_{12} = -\int_0^\infty \{[f'(\eta)+g'(\eta)]\cosh(\eta x_1)+\eta x_1 g'(\eta)\sinh(\eta x_1)\}\cos(\eta x_2)\,\mathrm{d}\eta$$
$$+\int_0^\infty \{[1-2\nu'-sx_2]\phi'(s)+[-1-sx_2]s\omega'(s)\}\exp(sx_2)\cos(sx_1)\,\mathrm{d}s.$$

This solution is obtained in [2] from the superposition of infinite strip and half-plane solutions. The corresponding solution S'' can be inferred from S' in (4) by the following replacements

$$(x_1, x_2, u'_1, u'_2, \tau'_{11}, \tau'_{12}, \tau'_{22}, f', g', \phi', \omega', \nu', \mu') \to$$
$$(-x_1, -x_2, -u''_1, -u''_2, \tau''_{11}, \tau''_{12}, \tau''_{22}, f'', g'', \phi'', \omega'', \mu''). \qquad (5)$$

The lateral boundary conditions $(1)_2$ and $(1)_3$ will be satisfied provided f', g' (f'', g'') are given in terms of ϕ', ω' (ϕ'', ω'') by

$$f(\eta) = \{L_1(\phi,\omega)(\eta)[\cosh(\eta h)+\eta h\sinh(\eta h)] - L_2(\phi,\omega)(\eta)[\eta h\cosh(\eta h)]\}$$
$$\times[\sinh(\eta h)\cosh(\eta h)+\eta h]^{-1},$$
$$g(\eta) = \{-L_1(\phi,\omega)(\eta)[\cosh(\eta h)] + L_2(\phi,\omega)(\eta)[\sinh(\eta h)]\}[\sinh(\eta h)\cosh(\eta h)+\eta h]^{-1} \qquad (6)$$

in which

$$L_1(\phi,\omega)(\eta) = -\frac{4}{\pi}\int_0^\infty \{[\nu\eta - s^2\eta(\eta^2+s^2)^{-1}]\phi(s)+[\eta^3 s(\eta^2+s^2)^{-1}]\omega(s)\}\sin(sh)(\eta^2+s^2)^{-1}\,\mathrm{d}s,$$
$$L_2(\phi,\omega)(\eta) = -\frac{4}{\pi}\int_0^\infty \{[-(1-\nu)s+\eta^2 s(\eta^2+s^2)^{-1}]\phi(s)+\eta^2 s^2(\eta^2+s^2)^{-1}\omega(s)\}\cos(sh)$$
$$\times(\eta^2+s^2)^{-1}\,\mathrm{d}s. \qquad (7)$$

Equations (6) and (7) are valid for (′) or (″) attached to all of f, g, ϕ, ω, ν. Finally ϕ, ω are related to $u_1(x_1, 0)$ and $\tau_{22}(x_1, 0)$ by

$$\phi(s) = \int_0^\infty \Phi(x_1)\sin(sx_1)\,\mathrm{d}x_1, \quad \omega(s) = \int_0^\infty \Omega(x_1)\cos(sx_1)\,\mathrm{d}x_1 \qquad (8)$$

where

$$\Phi(x_1) = \mp[\sqrt{(2/\pi)}/4\mu(1-\nu)]\tau_{22}(x_1,0), \quad \Omega(x_2) = \pm[\sqrt{(2/\pi)}/2(1-\nu)]u_1(x_1,0), \qquad (9)$$

in which the upper signs are used for S'' and the lower ones for S'.

Notice that because of (8), (9), the semi-infinite strip solutions S', S'' accommodate the mixed end boundary conditions of odd normal traction and even tangential displacement. It can also be verified that (4) satisfies the conditions

$$\tau_{ij}\to 0 \quad \text{as} \quad x_2\to\pm\infty, \qquad i,j=1,2,$$
$$\int_{-h}^{h}\tau_{i2}(x_1,0)\,\mathrm{d}x_1 = 0, \quad i=1,2, \qquad \int_{-h}^{h} x_1\tau_{22}(x_1,0)\,\mathrm{d}x_1 = 0. \qquad (10)$$

Without loss in generality we fix infinitesimal rigid displacements such that

$$u_1(\pm h, 0) = 0. \qquad (11)$$

In addition to the boundary functions in (9) we need for (2) the functions $u_2(x_1,0)$ and $\tau_{12}(x_1,0)$. If we define ϕ_1, ϕ_2 in terms of Φ, Ω by

$$\phi_1(x_1) = 2\mu\sqrt{(2/\pi)}\Phi(x_1) = \mp[\pi(1-\nu)]^{-1}\tau_{22}(x_1,0),$$
$$\phi_2(x_1) = 2\mu\sqrt{(2/\pi)}(\mathrm{d}/\mathrm{d}x_1)\Omega(x_1) = \pm[2\mu/\pi(1-\nu)](\mathrm{d}/\mathrm{d}x_1)u_1(x_1,0), \qquad (12)$$

then it follows from (4)–(9), (12) that (see [2] for details)

$$g_i(x_1)=\sum_{j=1}^{2}\int_{-h}^{h}\left[\frac{a_{ij}}{t-x_1}+k_{ij}(x_1,t)\right]\phi_j(t)\,\mathrm{d}t,\quad i=1,2,\tag{13}$$

in which

$$g_1(x_1)=2\mu(\mathrm{d}/\mathrm{d}x_1)u_2(x_1,0),\quad g_2(x_1)=\tau_{12}(x_1,0),\tag{14}$$

and a_{ij}, k_{ij} are defined by

$$\begin{gathered}a_{11}=(3-4\nu)/2,\quad a_{12}=-a_{21}=-(1-2\nu)/2,\quad a_{22}=1/2,\\ hk_{1i}(x_1,t)=A_i(\nu,\nu,y,\tau),\quad hk_{2i}(x_1,t)=A_i(\nu,1,y,\tau),\quad i=1,2,\\ y=x_1/h,\quad \tau=t/h,\end{gathered}\tag{15}$$

and A_i are defined (for $p=1$ or ν) by

$$\begin{aligned}A_i(\nu,p,y,\tau)=\frac{1}{2}\int_0^{\infty}&[A_{i1}(\nu,p,\tau,\xi)\cosh(\xi y)+A_{i2}(\nu,p,\tau,\xi)\xi y\sinh(\xi y)]\\ &\times[\sinh\xi\cosh\xi+\xi]^{-1}\exp[-\xi(1-\tau)]\,\mathrm{d}\xi\end{aligned}\tag{16}$$

with

$$\begin{aligned}A_{11}(\nu,p,\tau,\xi)&=-[1-2\nu-\xi(1-\tau)](\cosh\xi+\xi\sinh\xi)-[2-2\nu-\xi(1-\tau)]\xi\cosh\xi\\ &\quad+(3-2p)[2-2\nu-\xi(1-\tau)]\sinh\xi+(3-2p)[1-2\nu-\xi(1-\tau)]\cosh\xi,\\ A_{12}(\nu,p,\tau,\xi)&=[2-2\nu-\xi(1-\tau)]\sinh\xi+[1-2\nu-\xi(1-\tau)]\cosh\xi,\\ A_{21}(\nu,p,\tau,\xi)&=-[1+\xi(1-\tau)](\cosh\xi+\xi\sinh\xi)-\xi(1-\tau)\xi\cosh\xi\\ &\quad+(3-2p)[1+\xi(1-\tau)]\cosh\xi+(3-2p)\xi(1-\tau)\sinh\xi,\\ A_{22}(\nu,p,\tau,\xi)&=[1+\xi(1-\tau)]\cosh\xi+\xi(1-\tau)\sinh\xi.\end{aligned}\tag{17}$$

Notice that if τ_{22}, u_1 are known at $x_2=0$, so that ϕ_1, ϕ_2 are known, eqn (13) gives $(\mathrm{d}/\mathrm{d}x_1)u_2$ and τ_{12} there. Conversely, if u_2, τ_{12} are known at $x_2=0$, so that g_1, g_2 are known, (13) represents a system of integral equations for determining $(\mathrm{d}/\mathrm{d}x_1)u_1$ and τ_{22} there.

REDUCTION TO A SYSTEM OF INTEGRAL EQUATIONS

Next we determine Φ', Ω', Φ'', Ω'' in the solutions S', S'' so that $(1)_1$, $(1)_4$ and (2) are satisfied. The superposition is

$$\hat{S}''=\mathring{S}''+S'',\quad \hat{S}'=\mathring{S}'+S',\tag{18}$$

with $\mathring{S}''$, $\mathring{S}'$ given by (3) and S'', S' given by (4)–(9). In view of (10), (3), the solutions $\hat{S}''$, $\hat{S}'$ given by (18) satisfy $(1)_1$, $(1)_4$, so all that remains is to satisfy (2). With the use of (3), (12)–(14) and (18), it follows that (2) requires

$$\begin{gathered}\phi_2''(x_1)=-\frac{\mu''(1-\nu')}{\mu'(1-\nu'')}\phi_2'(x_1)-\frac{2\mu''}{\pi(1-\nu'')}\left(\frac{3M}{4h^3}\right)\left(\frac{\nu''}{\mu''}-\frac{\nu'}{\mu'}\right)x_1,\\ \phi_1''(x_1)=-[(1-\nu')/(1-\nu'')]\phi_1'(x_1),\\ \sum_{j=1}^{2}\int_{-h}^{h}\left\{\left[\frac{a_{1j}''}{t-x_1}+k_{1j}''(x_1,t)\right]\frac{\phi_j''(t)}{\mu''}-\left[\frac{a_{1j}'}{t-x_1}+k_{1j}'(x_1,t)\right]\frac{\phi_j'(t)}{\mu'}\right\}\mathrm{d}t=0,\\ \sum_{j=1}^{2}\int_{-h}^{h}\left\{\left[\frac{a_{2j}''}{t-x_1}+k_{2j}''(x_1,t)\right]\phi_j''(t)-\left[\frac{a_{2j}'}{t-x_1}+k_{2j}'(x_1,t)\right]\phi_j'(t)\right\}\mathrm{d}t=0,\end{gathered}\tag{19}$$

which gives four equations for ϕ_j', ϕ_j'', $j=1,2$, or, from (12) for Φ', Φ'', $(\mathrm{d}/\mathrm{d}x_1)\Omega'$, $(\mathrm{d}/\mathrm{d}x_1)\Omega''$. The first two of (19) allow us to eliminate ϕ_i'' from the last two of (19) to arrive at the system of two equations

$$\sum_{j=1}^{2} \int_{-h}^{h} \left[\frac{b_{ij}}{t - x_1} + l_{ij}(x_1, t)\right] \phi_j'(t)\,\mathrm{d}t = f_i(x_1), \tag{20}$$

in which

$$\begin{aligned}
b_{11} &= a''_{11} k(1-\nu')/(1-\nu'') + a'_{11},\\
b_{12} &= a''_{12}(1-\nu')/(1-\nu'') + a'_{12}, \quad b_{21} = -k b_{12},\\
b_{22} &= a''_{22}(1-\nu')/(1-\nu'') + k a'_{12}, \qquad k = \mu'/\mu'',\\
l_{11} &= [k''_{11} k(1-\nu')/(1-\nu'') + k'_{11}],\\
l_{12} &= [k''_{12}(1-\nu')/(1-\nu'') + k'_{12}],\\
l_{21} &= [k''_{21} k(1-\nu')/(1-\nu'') + k k'_{21}],\\
l_{22} &= [k''_{22}(1-\nu')/(1-\nu'') + k k'_{22}],\\
f_i(x_1) &= -\hat{M} \int_{-h}^{h} \left[\frac{a''_{i2}}{t - x_1} + k''_{i2}(x_1, t)\right] t\,\mathrm{d}t,\\
\hat{M} &= [2/\pi(1-\nu'')](3M/4h^3)(k\nu'' - \nu').
\end{aligned} \tag{21}$$

We observe formally that f_i vanishes for material combinations such that $\mu'\nu'' = \mu''\nu'$. From (20) this implies that the ϕ_i' vanish if there are no eigenfunctions, and (19) then implies ϕ_i'' also vanish. It follows from (12) that Φ, Ω vanish and hence (6) predicts that S', S'' vanish. For these combinations $\mathring{S}', \mathring{S}''$ in (3) supply the entire solution. In general the f_i in (21) do not vanish and, as will be apparent later on, the integrals define functions with logarithmic singularities at $x_1 = \pm h$. These singularities must be removed before (20) can be solved numerically.

ANALYSIS OF THE SINGULAR INTEGRAL EQUATIONS

The integral eqns (20) are singular as is apparent from the Cauchy kernels. The kernels l_{ij} are also singular due to the behavior of the integrands in (16) as $\xi \to \infty$. Also, the integrands in (16) have a first order singularity at $\xi = 0$ that must be removed. In the manner of [1, 2] we use the condition

$$\int_{-h}^{h} \phi_i(t)\,\mathrm{d}t = 0, \quad i = 1, 2 \tag{22}$$

to replace A_i in (16) by

$$\begin{aligned}
A_i &= \int_0^\infty a_i\,\mathrm{d}\xi = A_i^\infty + A_i^R,\\
A_i^R &= \int_0^1 (a_i - a_i^\infty - a_i^0)\,\mathrm{d}\xi + \int_1^\infty (a_i - a_i^\infty)\,\mathrm{d}\xi + \frac{d_i}{2} U,\\
a_1^\infty &= \{[-1 + 2\nu + (3-4\nu)(3 - 2p - \xi) - (5 - 4p - 2\xi)\xi(1-\tau)] \cosh(\xi y)\\
&\quad + [3 - 4\nu - 2\xi(1-\tau)]\xi y \sinh(\xi y)\} \exp[-\xi(2-\tau)],\\
a_2^\infty &= \{[2 - 2p - \xi + (5 - 4p - 2\xi)\xi(1-\tau)] \cosh(\xi y)\\
&\quad + [1 + 2\xi(1-\tau)]\xi y \sinh(\xi y)\} \exp[-\xi(2-\tau)], \quad i = 1, 2,\\
a_i^0 &= d_i \xi^{-1} \cosh(\xi y) \exp[-\xi(1-\tau)], \quad i = 1, 2,\\
d_1 &= (1-p)(1-2\nu)/2, \quad d_2 = (1-p)/2,\\
U(y, \tau) &= \sum_{n=1}^{\infty} (-1)^n \frac{(1-\tau-y)^n}{n\,n!} + \sum_{n=1}^{\infty} (-1)^n \frac{(1-\tau+y)^n}{n\,n!},
\end{aligned} \tag{23}$$

$$A_i^\infty = \int_0^\infty a_i^\infty \, \mathrm{d}\xi,$$

$$A_1^\infty = [3/2 + 4\nu p - 5\nu - p + (3 + 2\nu - 2p)(1 - y)(\mathrm{d}/\mathrm{d}y) - (1 - y)^2(\mathrm{d}^2/\mathrm{d}y^2)](2 - \tau - y)^{-1}$$
$$+ [3/2 + 4\nu p - 5\nu - p - (3 + 2\nu - 2p)(1 + y)(\mathrm{d}/\mathrm{d}y) - (1 + y)^2(\mathrm{d}^2/\mathrm{d}y^2)](2 - \tau + y)^{-1},$$

$$A_2^\infty = [7/2 - 3p - (5 - 2p)(1 - y)(\mathrm{d}/\mathrm{d}y) + (1 - y)^2(\mathrm{d}^2/\mathrm{d}y^2)](2 - \tau - y)^{-1}$$
$$+ [7/2 - 3p + (5 - 2p)(1 + y)(\mathrm{d}/\mathrm{d}y) + (1 + y)^2(\mathrm{d}^2/\mathrm{d}y^2)](2 - \tau + y)^{-1}.$$

The integrals and series in (23) converge for all values of (y, τ) in $[-1, 1] \times [-1, 1]$ and define regular kernels A_i^R. The singular parts of A_i are given by A_i^∞. In view of (20), (21), (15), (16) we can decompose the kernels l_{ij} in (20) in the same way, i.e.

$$l_{ij} = l_{ij}^R + l_{ij}^\infty. \tag{24}$$

In order to determine the singularities in the solutions ϕ'_j of (20) we use the methods of [3] and write (20) as

$$\sum_{j=1}^{2} \int_{-h}^{h} \left[\frac{b_{ij}}{t - x_1} + l_{ij}^\infty(x_1, t) \right] \phi'_j(t) \, \mathrm{d}t = B_i(x_1) \tag{25}$$

where

$$B_i(x_1) = f_i(x_1) - \sum_{j=1}^{2} \int_{-h}^{h} l_{ij}^R(x_1, t) \phi'_j(t) \, \mathrm{d}t, \tag{26}$$

so that $B_i(x_1)$ contains at most logarithmic singularities on $|x_1| \leqq h$ provided the singularities in ϕ'_j are integrable. According to [3, Chap. 4] we define $H_j(t)$ through

$$\phi'_j(t) = H_j(t)/(h^2 - t^2)^{\gamma_j}, \quad |t| < h \tag{27}$$

where $Re(\gamma_j) < 1$ and $H_j(t)$ satisfies a Holder condition. The appropriate singular integral analysis leads to the following condition for determining the indices γ_j in (27);

$$\begin{aligned} \gamma_1 &= \gamma_2 = \gamma, \\ \Delta(\gamma) &= \det[-b_{ij} \cos(\pi\gamma) + n_{ij}(\gamma)] = 0, \end{aligned} \tag{28}$$

where

$$\begin{aligned} n_{11} &= M_1(\nu'', \nu'', \gamma) k(1 - \nu')/(1 - \nu'') + M_1(\nu', \nu', \gamma), \\ n_{12} &= M_2(\nu'', \nu'', \gamma)(1 - \nu')/(1 - \nu'') + M_2(\nu', \nu', \gamma), \\ n_{21} &= M_1(\nu'', 1, \gamma) k(1 - \nu')/(1 - \nu'') + k M_1(\nu', 1, \gamma), \\ n_{22} &= M_2(\nu'', 1, \gamma)(1 - \nu')/(1 - \nu')/(1 - \nu'') + k M_2(\nu', 1, \gamma), \end{aligned} \tag{29}$$

in which (for $p = 1, \nu$)

$$\begin{aligned} M_1(\nu, p, \gamma) &= 3/2 - 5\nu - p + 4p\nu + (3 + 2\nu - 2p)\gamma - \gamma(\gamma + 1), \\ M_2(\nu, p, \gamma) &= 7/2 - 3p - (5 - 2p)\gamma + \gamma + 1). \end{aligned} \tag{30}$$

According to (21) and (28)–(30) γ apparently depends on the elastic constants ν', ν'' and $k = \mu'/\mu''$. We expect from [1], [5] that this dependence should be expressible in terms of the two composite parameters α, β introduced in [4] and defined for plane strain by

$$\alpha = \frac{k(1 - \nu'') - (1 - \nu')}{k(1 - \nu'') + (1 - \nu')}, \quad \beta = \frac{k(1 - 2\nu'') - (1 - 2\nu')}{2k(1 - \nu'') + 2(1 - \nu')}. \tag{31}$$

It can be verified from (21) and (28)–(31) that

$$\Delta(\gamma) = -4[(1-\nu')/(1-\nu'')][k(1-\nu'')+(1-\nu')]^2 D(\alpha,\beta;\gamma-2) \tag{32}$$

where $D(\alpha, \beta;\ \gamma-2)$ is given in [5] as

$$D(\alpha,\beta;\gamma-2) = [\cos^2(\gamma\pi/2)-(1-\gamma)^2]^2\beta^2 + 2(1-\gamma)^2[\cos^2(\gamma\pi/2)-(1-\gamma)^2]\alpha\beta + (1-\gamma)^2[(1-\gamma)^2-1]\alpha^2 + \cos^2(\gamma\pi/2)\sin^2(\gamma\pi/2), \tag{33}$$

and since the coefficient of D in (32) cannot vanish for $0 \leqq \nu \leqq \frac{1}{2}$, $0 \leqq k < \infty$ it follows that the solutions γ of (28) are also given by

$$D(\alpha,\beta;\gamma-2) = 0. \tag{34}$$

The values of γ satisfying (34) and $Re(\gamma) < 1$ for all physically relevant values of α, β were given in [1] and are reproduced here in Fig. 2. As observed in [1] γ is real and satisfies $0 < \gamma < 1$ for $\alpha(\alpha-2\beta) > 0$, $\gamma = 0$ for $\alpha(\alpha-2\beta) = 0$ and $\gamma < 0$ for $\alpha(\alpha-2\beta) < 0$. Therefore the functions $\phi_i'(t)$ are bounded at the singular points $t = \pm h$ for $\alpha(\alpha-2\beta) \leqq 0$. A numerical technique for obtaining approximate solutions of (20) can be found in [6] for the case when the right hand side is bounded. We can use this technique with slight modification to solve (20) in the presence of the logarithmically singular f_i.

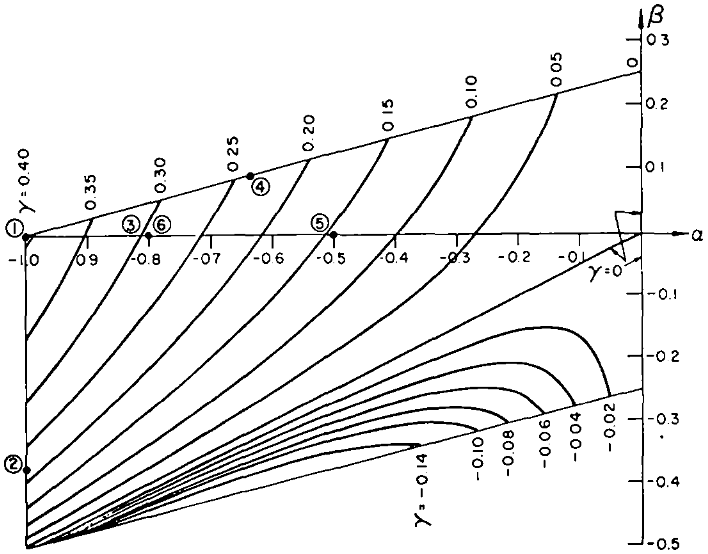

Fig. 2. Dependence of index γ on composite parameters α, β.

REMOVAL OF THE LOGARITHMIC SINGULARITIES

Before we can apply the quadrature method in [6] to eqn (20) we must remove the logarithmic singularities from the functions f_i. First we show under what conditions these singularities arise. For this we need the integral expressions

$$\int_{-1}^{1} \frac{\tau\, d\tau}{\tau - y} = 2 + y \ln\left(\frac{1-y}{1+y}\right),$$

$$\sum_{k=0}^{2} \int_{-1}^{1} C^{(k)}(\mp 1)^k (1 \mp y)^k \frac{d^k}{dy^k}(2-\tau \mp y)^{-1}\tau\, d\tau = 4C^{(2)}(3\mp y)^{-2} + C^{(1)}\left[2\left(\frac{2\mp y}{3\mp y}\right) + (1\mp y)\ln\left(\frac{1\mp y}{3\mp y}\right)\right] - C^{(0)}\left[2 + (2\mp y)\ln\left(\frac{1\mp y}{3\mp y}\right)\right]. \tag{35}$$

From (15), (23) and the above integrals we obtain

$$\int_{-h}^{h}\left[\frac{a_{ij}}{t-x_1}+k_{ij}(x_1,t)\right]t\,dt=\int_{-h}^{h}k_{ij}^{R}(x_1,t)\,t\,dt+\bar{f}_{ij}(x_1)+\tilde{f}_{ij}(x_1), \tag{36}$$

where

$$\begin{aligned}\bar{f}_{ij}(x_i)/h &= 2a_{ij}+4C_{ij}^{(2)}[(3-y)^{-2}+(3+y)^{-2}]+C_{ij}^{(1)}\left[4\left(\frac{6-y^2}{9-y^2}\right)\right.\\ &\quad\left.+(1-y)\ln\left(\frac{1-y}{3-y}\right)+(1+y)\ln\left(\frac{1+y}{3+y}\right)\right]\\ &\quad -C_{ij}^{(0)}[4-(2-y)\ln(3-y)-(2+y)\ln(3+y)], \end{aligned} \tag{37}$$

$$\tilde{f}_{ij}(x_i)/h = a_{ij}y\ln\left(\frac{1-y}{1+y}\right)-C_{ij}^{(0)}[(2-y)\ln(1-y)+(2+y)\ln(1+y)]$$

in which the coefficients $C_{ij}^{(k)}$ can be obtained by comparing (15), (23) with (35), (36). We see that $\bar{f}_{ij}$ is bounded in $|y|\le 1$, but $\tilde{f}_{ij}$ has logarithmic singularities unless $C_{ij}^{(0)}=a_{ij}$. Since $C_{12}^{(0)}=-3\nu+7/2$, $a_{12}=\nu-1/2$, and $C_{22}^{(0)}=a_{22}=1/2$ it follows that f_1 in (20) is singular whereas f_2 is bounded.

In order to remove the logarithmic singularity from f_1 we define new unknowns $\bar{\phi}_i$ in (20) by

$$\bar{\phi}_1=\phi_1',\quad \bar{\phi}_2=\phi_2'-Ft \tag{38}$$

where F is a constant to be determined. With (38) we can write (20) as

$$\sum_{j=1}^{2}\int_{-h}^{h}\left[\frac{b_{ij}}{t-x_1}+l_{ij}(x_1,t)\right]\bar{\phi}_j(t)\,dt=\hat{f}_i(x_1) \tag{39}$$

where

$$\begin{aligned}\hat{f}_1(x_1)&=-[\hat{M}+F(1-\nu')/(1-\nu'')]\left[\int_{-h}^{h}k_{12}^{R''}(x_1,t)t\,dt+\bar{f}_{12}''(x_1)+\tilde{f}_{12}''(x_1)\right]\\ &\quad -F\left[\int_{-h}^{h}k_{12}^{R'}(x_1,t)t\,dt+\bar{f}_{12}'(x_1)+\tilde{f}_{12}'(x_1)\right],\end{aligned} \tag{40}$$

$$\hat{f}_2(x_1)=-\frac{F}{2}\left[\int_{-h}^{h}k_{22}^{R''}(x_1,t)t\,dt+\bar{f}_{22}''(x_1)+\tilde{f}_{22}''(x_1)\right]-kF\left[\int_{-h}^{h}k_{22}^{R'}(x_1,t)t\,dt+\bar{f}_{22}'(x_1)+\tilde{f}_{22}'(x_1)\right].$$

With use of (37) in (40) it is apparent that $\hat{f}_i$ are bounded on $|x_1|\le h$ provided

$$F=-\hat{M}(1-\nu'')/2(1-\nu'). \tag{41}$$

Thus the system (39), with unknowns defined in (38), has a bounded right hand side (40) when F is given by (41).

NUMERICAL SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

By use of the numerical methods in [6] the singular integrals in (39) can be approximated directly by quadrature methods. The quadrature formula to be utilized depends on the index γ appropriate to the functions ϕ_j'. For the values determined by (34) the Gauss–Jacobi formula must be used. The integrals in $\hat{f}_i$ in (40) can be evaluated by use of Simpson's rule.

In terms of dimensionless variables y, τ, defined in (15), eqn (39) appears as

$$\sum_{j=1}^{2}\int_{-1}^{1}\left[\frac{b_{ij}}{\tau-y}+hl_{ij}(hy,h\tau)\right]\frac{\psi_j(\tau)\,d\tau}{(1-\tau^2)^{\gamma}}=\hat{f}_i(hy), \tag{42}$$

in which

$$\bar{\psi}_i(\tau) = \bar{\phi}_i(t)(1-\tau^2)^\gamma. \tag{43}$$

The algebraic system of order $2N$ corresponding to (42), (22) is

$$\begin{aligned}\sum_{j=1}^{2}\sum_{J=1}^{N}\left[\frac{b_{ij}}{\tau_J - y_K} + hl_{ij}(hy_K, h\tau_J)\right]A_J\bar{\psi}_j(\tau_J) &= \hat{f}_i(hy_K), \quad i=1,2,\\ \sum_{J=1}^{N} A_J\bar{\psi}_i(\tau_J) &= 0, \qquad\quad i=1,2,\end{aligned} \tag{44}$$

where the latter equations results from (22), (38) and it must be appended to complete the system, since $J=1,2,\ldots,N$ but $K=1,2,\ldots,N-1$ when τ_J, y_K, A_J are determined in the Gauss–Jacobi scheme, i.e.

$$\begin{aligned}A_J &= -\frac{2(N-\gamma+1)}{(N+1)!(N-2\gamma+1)}\frac{\Gamma^2(N-\gamma+1)}{\Gamma(N-2\gamma+1)}\frac{2^{-2\gamma}}{P_N'^{(-\gamma,-\gamma)}(\tau_J)P_{N+1}^{(-\gamma,-\gamma)}(\tau_J)},\\ P_N^{(\gamma,\gamma)}(\tau) &= \frac{\Gamma(\gamma+N+1)}{\Gamma(2\gamma+N+1)}\sum_{k=0}^{N}\frac{\Gamma(2\gamma+N+k+1)(\tau-1)^k}{k!(N-k)!2^k\Gamma(\gamma+k+1)},\\ \tau_J,\ J &= 1,2,\ldots,N \quad \text{are } N \text{ roots of } P_N^{(-\gamma,-\gamma)}(\tau)=0,\\ y_K,\ K &= 1,2,\ldots,N-1 \quad \text{are } N-1 \text{ roots of } P_{N-1}^{(1-\gamma,1-\gamma)}(y)=0.\end{aligned} \tag{45}$$

NUMERICAL RESULTS AND DISCUSSION

The system (44) was solved numerically for six different composites, which are identified by their elastic constants k, ν', ν'' in Fig. 3. Also given in the table are the corresponding composite parameters α, β as computed from (31) and the index γ as obtained from Fig. 2 with these values of α, β. All of the composites chosen give positive values of γ.

After the bounded functions $\bar{\psi}_i$ were obtained we computed $\tau'_{22}(x_1, 0)$, $\tau'_{12}(x_1, 0)$ and $\mathrm{d}u'_2/\mathrm{d}x_1$ at $x_2 = 0$. From (12), (43)

$$\tau'_{22}(h\tau_J, 0) = \pi(1-\nu')\psi_1(\tau_J)/(1-\tau_J^2)^\gamma, \tag{46}$$

while (13), (14) and the quadrature procedure give

$$\begin{aligned}2\mu'(\mathrm{d}/\mathrm{d}x_1)u_2(hy_K, 0) &= \sum_{j=1}^{2}\sum_{J=1}^{N}\left[\frac{a'_{1j}}{\tau_J - y_K} + hk'_{1j}(hy_K, h\tau_J)\right]A_J\psi_j(\tau_J),\\ \tau'_{12}(hy_K, 0) &= \sum_{j=1}^{2}\sum_{J=1}^{N}\left[\frac{a'_{2j}}{\tau_J - y_K} + hk'_{2j}(hy_K, h\tau_J)\right]A_J\psi_j(\tau_J).\end{aligned} \tag{47}$$

	k	ν'	ν''	α	β	γ
①	0	.5	—	-1	0	.410
②	0	.2	—	-1	-.375	.221
③	.111	.5	.5	-.8	0	.295
④	.111	.5	0	-.636	-.091	.240
⑤	.333	.5	.5	-.5	0	.145
⑥	.059	.47	0	-.8	0	.295

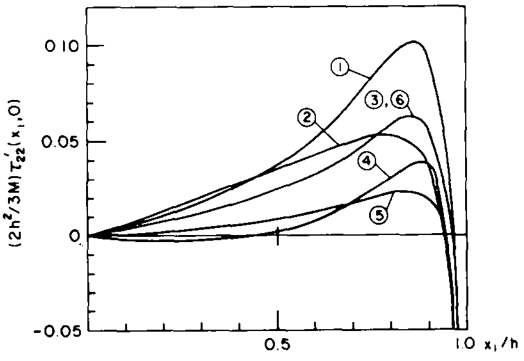

Fig. 3. Interface normal stress for various composites.

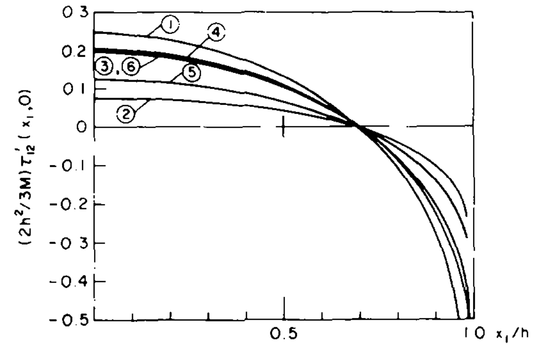

Fig. 4. Interface shear stress for various composites.

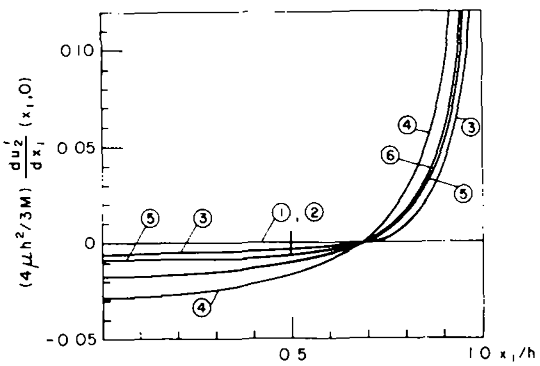

Fig. 5. Interface slope for various composites.

Figure 3 gives the residual interface normal stress $(2h^2/3M)\tau'_{22}(x_1,0)$, Fig. 4 shows $(2h^2/3M)\hat{\tau}_{12}(x_1,0)$ and Fig. 5 displays the interface slope $(4h^2\mu/3M)(d/dx_1)\hat{u}_2(x_1,0)$.

Next we observe that the solution obtained here should agree with the clamped semi-infinite strip bending solution in [3] in the limit when the upper semi-strip becomes rigid. When $\mu'' \to \infty$ so that $k \to 0$, and by (31) $\alpha \to -1$, we obtain from the second of (20) with (21)

$$\int_{-h}^{h}\left[\frac{a''_{22}}{t-x_1}+k''_{22}(x_1,t)\right]\phi'_2(t)\,dt = \frac{\nu'}{\pi(1-\nu')}\left(\frac{3M}{2h^3}\right)\int_{-h}^{h}\left[\frac{a''_{22}}{t-x_1}+k''_{22}(x_1,t)\right]t\,dt \tag{48}$$

from which there follows

$$\phi'_2(t) = \nu'[\pi(1-\nu')]^{-1}(3M/2h^2)t. \tag{49}$$

With this the first of (20) yields

$$\int_{-h}^{h}\left[\frac{a'_{11}}{t-x_1}+k'_{11}(x_1,t)\right]\phi'_1(t)\,dt = -\frac{\nu'}{\pi(1-\nu')}\left(\frac{3M}{2h^3}\right)\int_{-h}^{h}\left[\frac{a'_{12}}{t-x_1}+k'_{12}(x_1,t)\right]t\,dt, \tag{50}$$

for determining ϕ'_1. Equations (49), (50) agree with (45) of [2]. Furthermore, the index γ in Fig. 2 of [2] corresponds to γ in Fig. 2 at $\alpha = -1$. Hence the reduction is complete. The right hand side of (50) has a logarithmic singularity which can be removed by replacing ϕ'_1 by $\bar{\phi}_1 + Ft$ and determining F in the same manner as above.

Finally, we observe from Figs. 3 and 4 that the interface traction is the same for both composites 3 and 6, as it must be since the α, β are identical in both cases (see [4]). The same is not true of the interface slope in Fig. 5.

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